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1. Boundary value problems and functional equations

1.1. Scalar Riemann-Hilbert problem

The results of Chapter 4 [MiR] are briefly presented in this section.

Let us consider mutually disjointed disks $D_k := \{z \in \mathbb{C} : \ |z - a_k| < r_k\}$ $(k = 0, 1, ..., n)$ in the complex plane C. Let $D := \overset{\wedge}{\mathbb{C}} \setminus \cup_{k=0}^{n} (D_k \cup \partial D_k)$, $\overset{\wedge}{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. We assume the circles $\partial D_k := \{t \in \mathbb{C} : \ |t - a_k| = r_k\}$ are orientated in a clockwise sense. Given $\lambda_k(t)$, $f_k(t)$ as Hölder continuous functions on ∂D, $\lambda_k(t) \neq 0$. Find a function $\varphi(z)$ analytic in D continuous in $D \cup \partial D$ with the following boundary condition

$$Re\overline{\lambda_k(t)}\varphi(t) = f_k(t), \ |t - a_k| = r_k, \ k = 0, 1, ..., n. \tag{1}$$

The problem (1) is called the *(Riemann-) Hilbert boundary value problem.* This problem has been discussed in classical books [gak, mus]. One can find there the solution of (1) in closed form for simple and double connected domains ($n = 0$ or $n = 1$). Solvability of the Hilbert problem is discussed in [bw, gak, mus, vek, zv] and other works by a method of integral equations.

I have solved the scalar Riemann-Hilbert problem for any multiply connected domain *in closed form.* The problem (1) is closely related to harmonic measures of the domain D, classical Dirichlet problem, the Schwarz operator and so forth. The functions $\alpha_s(z)$ $(s = 0, 1, ..., n)$ is called *the harmonic measures* of the domain D if it is harmonic in D, continuous in $D \cup \partial D$ and satisfies the boundary conditions

$$\alpha_s(t) = \delta_{sk}, \ |t - a_k| = r_k, \ k = 0, 1, ..., n, \tag{2}$$

where δ_{sk} is the Kronecker symbol. I present here only solution to the problem (2).

Consider inversions with respect to the circles $|z - a_k| = r_k$ and their compositions

$$z^*_{(k)} := \frac{r_k^2}{\overline{z - a_k}} + a_k, \ z^*_{(k_j k_{j-1} \dots k_1)} := \left(z^*_{(k_{j-1} \dots k_1)}\right)^*_{(k_j)}. \tag{3}$$

In the sequence $k_1, k_2, ..., k_j$ no two neighboring numbers are equal. When j is even, these are Möbius transformations in z. If j is odd, we have transformations in $\overline{z}$. Hence, the mappings can be written in the form

$$\begin{aligned} \gamma_j(z) &= (e_j z + b_j) / (c_j z + d_j), \ j \in 2\mathbb{Z}, \\ \gamma_j(\overline{z}) &= (e_j \overline{z} + b_j) / (c_j \overline{z} + d_j), \ j \in 2\mathbb{Z} + 1, \end{aligned} \tag{4}$$

where $e_jd_j - b_jc_j = 1$. Here $\gamma_0(z) := z$ (identical mapping),

$$\gamma_1(\overline{z}) := z^*_{(1)} \ , ..., \ \gamma_n(\overline{z}) := z^*_{(n)}$$

(n simple inversions)

$$\gamma_{n+1}(z) := z^*_{(12)}, \gamma_{n+2}(z) := z^*_{(13)}, ..., \gamma_{n^2}(z) := z^*_{(n,n-1)}$$

($n^2 - n$ pairs of inversions), $\gamma_{n^2+1}(\overline{z}) := z^*_{(121)}, ...$ and so on. The number j is called the level of the mapping γ_j. The indexes j of γ_j are ordered in such a way that the level is increasing. The functions (3) or (4) generate a Schottky group $\mathcal{K}$. Thus, each element of $\mathcal{K}$ is presented in the form of the composition of inversions (3) or in the form of linearly ordered functions (4). Let $\mathcal{K}_m$ be such a subset of $\mathcal{K}\backslash\{\gamma_0\}$ that the last inversion of each element of $\mathcal{K}_m$ is different from $z^*_{(m)}$, i.e., $\mathcal{K}_m = \{z^*_{(k_j k_{j-1} \ldots k_1)} : \ k_j \neq m\}$. Let us fix a point $z_0 \in D\backslash \{\infty\}$ and introduce the functions

$$\psi_m(z) = \ln \left[\prod_{\gamma_j \in \mathcal{K}_m} \psi_m^{(j)}(z) \right], \tag{5}$$

where we have respectively for even level of $\gamma_j \in \mathcal{K}$

$$\psi_m^{(j)}(z) = \frac{\gamma_j(z) - a_m}{\gamma_j(z_0) - a_m}, \tag{6}$$

and for odd level of $\gamma_j \in \mathcal{K}$

$$\psi_m^{(j)}(z) = \overline{\frac{\gamma_j(\overline{z_0}) - a_m}{\gamma_j(\overline{z}) - a_m}}. \tag{7}$$

The infinite product (5) converges uniformly in every compact subset of $\overline{D}$ (see [MiR]). The multipliers in the infinite product (5) are ordered in accordance with the increasing level.

Let us fix k from the set $\{1, 2, ..., n\}$. The harmonic measure $\alpha_k(\zeta)$ associated to D satisfies the boundary conditions $\alpha_k(\zeta) = 1$, $|\zeta - a_k| = r_k$ and $\alpha_k(\zeta) = 0$, $|\zeta - a_m| = r_m$ for $m \neq k$. It is explicitly written in Theorem 4.10 of [MiR]

$$\alpha_k(\zeta) = \sum_{m=1}^{n} A_{km} Re[\psi_m(\zeta) + \ln(\zeta - a_m)] + A_k, \tag{8}$$

where $\psi_m(\zeta)$ has the form (5), the real constants A_{km} and A_k are described in Theorem 4.10 of [MiR].

Consider the function $\omega(z, \zeta)$ defined by the formulas (4.4.27) or (4.4.28) from [MiR]

$$\omega(z, \zeta) = \ln \prod_{j=1}^{\infty} \omega_j(z, \zeta), \tag{9}$$

where $\omega_j(z,\zeta)$ for odd levels has the form [MiR, p.146]

$$\omega_j(z,\zeta) = \frac{\overline{\zeta - \gamma_j(\overline{z_0})}}{\overline{\zeta - \gamma_j(\overline{z})}}. \tag{10}$$

If the level is odd $\omega_j(z,\zeta)$ has the form [MiR, p.146]

$$\omega_j(z,\zeta) = \frac{\zeta - \gamma_j(z)}{\zeta - \gamma_j(z_0)}. \tag{11}$$

The infinite product (9) converges uniformly on z in $\overline{D}$ for each fixed $\zeta \in D$ (uniformly on each compact subset of $\overline{D}\backslash\{\zeta\}$ if $\zeta \in \partial D$). Let $\mathcal{G} \subset \mathcal{K}$ be the subgroup consisting of γ_j of the even level, $\mathcal{F} \subset \mathcal{K}$ be the set of γ_j of the odd level.

Theorem 1 *The complex Green function $M(z,\zeta)$ associated to D is represented in the form*

$$M(z,\zeta) = \sum_{k=1}^{n} \alpha_k(\zeta)[\psi_k(z) + \ln(z - a_k)] - \omega(z,\zeta) - \ln(z-\zeta) + A(\zeta). \tag{12}$$

An operator solving the boundary value problem (1) with $\lambda_k\left(t\right) = 1$ and $f_k\left(t\right) = h_k\left(t\right) + c_k$, where $c_0 = 0$, c_k $(k = 1, 2, ..., n)$ are undetermined constants, is called the Shcwarz operator of D. Ultimately, the Shcwarz operator and solution to the Riemann–Hilbert problem have been also constructed in analytic form as the Green function.

1.2. R–linear problem

Given $a\left(t\right),\ b\left(t\right),\ c\left(t\right)$ as Hölder continuous functions on ∂D. Find a function $\varphi\left(z\right)$ analytic in D and $D^- := \cup_{k=0}^{n} D_k$, respectively, continuous in closures of the considered domains with the $\mathbb{R}$-linear conjugation condition

$$\varphi^+\left(t\right) = a\left(t\right)\varphi^-\left(t\right) + b\left(t\right)\overline{\varphi^-\left(t\right)} + c\left(t\right),\ t \in \partial D. \tag{13}$$

If $a\left(t\right) = 1$, $b\left(t\right) = \rho =$*constant*, $c\left(z\right)$ is analytic in $\cup_{k=0}^{n} D_k$, then (13) is a fundamental boundary value problem of the mechanics of composite materials (see Section 3). The solution of (13) for $|\rho| < 1$ in D has the form

$$\varphi\left(z\right) = \rho\sum_{k=0}^{n}\left[\overline{c\left(z_k^*\right)} + \overline{c\left(w_k^*\right)}\right] + \rho^2\sum_{k=0}^{n}\sum_{m\neq k}\left[c\left(z_{mk}^*\right) + c\left(w_{mk}^*\right)\right] + ...$$

1.3. Functional equations

The crucial points in solution to the problems (1) and (13) is to reduce them to functional equations. The simplest functional equation has the form [kucz]

$$\varphi\left(z\right) = G\left(z\right)\varphi\left[\alpha\left(z\right)\right] + g\left(z\right),\ |z| \leq r, \tag{14}$$

where known functions $G\left(z\right),\ g\left(z\right)$ and unknown function $\varphi\left(z\right)$ are meromorphic in $|z| < r$ and continuous in $|z| \leq r$. The given function $\alpha\left(z\right)$ maps conformally $|z| \leq r$ into $|z| < r$. Other functional equations are also discussed.

1.4. Poincaré series

Let $H(z)$ be a meromorphic function in the extended complex plane $\overset{\wedge}{\mathbb{C}}$. The Poincaré θ_{2q} -series [ak, ford]

$$\theta_{2q}(z) := \sum_{j=0}^{\infty} H(\gamma_j(z))(c_j z + d_j)^{-2q}, \quad (q \in \mathbb{Z}/2) \tag{15}$$

is associated with the group $\mathcal{K}$. Here $z \in B := \overset{\wedge}{\mathbb{C}} \setminus (B_1 \cup \Lambda(\mathcal{K}))$, B_1 is the set of poles of all $H(\gamma_j(z))$ and $\gamma_j(z)$, $\Lambda(\mathcal{K})$ is the limit set of $\mathcal{K}$. When $q > 1$ the series (15) converges absolutely and uniformly in every compact subset of B [ford]. When $q = 1$ the series (15) can be either absolutely convergent or absolutely divergent. It depends on the properties of $\mathcal{K}$. Necessary and sufficient conditions for absolute and uniform convergence of the series have been found in [ak] in terms of the Hausdorff dimension of $\Lambda(\mathcal{K})$. Let us note that absolute and uniform convergence was not studied separately in the previous papers.

Definition 1 *A point z is called a regular point of $\mathcal{K}$ if there exist numbers $k_1, k_2 ..., k_m$ such that $z^*_{k_m,...,k_1}$ belongs to $D \cup \partial D$.*

Theorem 2 *Let a rational function $H(z)$ has poles only at regular points of $\mathcal{K}$. Then the Poincaré θ_2-series converges uniformly in every compact subset of each region $(D)^*_{k_m,...,k_1} \cap B$. The order of summation depends on the region $(D)^*_{k_m,...,k_1}$.*

Construction of the Poincaré θ_2-series leads to solution to the Riemann–Hibert problem on Riemannian surfaces. For details see [Mityushev V. Riemann problem on double of multiply connected region, Annales Polon. Math., 1997, 77.1, 1-14] and [Mityushev V. On construction of Abelian differentials on the closed Riemannian surface, Slupskie Prace Mat.-Przyr. 11a, 213-228, 1997].

Key words of the other results

Queueing system analysis, heat equation, wave equation, fracture mechanics, reactive media, non-local problem, principal functionals, elastic plane problem, conformal map.

Conclusion

A new method of functional equations has been worked out. Some old and new problems have been solved by this method: the Riemann–Hilbert boundary value problem for multiply connected domains, uniform convergence of the Poincaré θ_2-series and the modified method of Schwarz, calculation of the transport properties of arrays of circular cylinders and others. Special attention is paid applications to composite materials, porous media and other applied problems (see next sections).

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