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3. Composite materials and porous media

Composite materials play an important role in creation of new materials. The well-know examples are given by reinforced concrete or fiber reinforced carbon composites. Typically, in such materials, the physical parameters are discontinuous and oscillate between the different values characterizing each of the components. The theory of homogenization is used to get a good approximation of the macroscopic behavior of such a heterogeneous material [G.W. Milton (2002), The Theory of Composites, Cambridge University Press].

This theory leads to a boundary value problem in the cell representing the composite. In the case of the steady process governed by the Laplace equation we arrive at the boundary value problems from Sections 1 and 2 in a class of doubly periodic functions. Let us represent a typical result of our study. An array of circular cylinders is imbedded in a homogeneous matrix material. The conductivity of the matrix is taken as unity and the conductivity of cylinders is expressed relative to this. The macroscopic behavior of the array of cylinders is derived by *the effective conductivity tensor* Λ_e. There are only approximate formulae (such as the Clausius-Mossotti approximation) or numerical algorithms to calculate Λ_e in the previous works. The method of functional equations allows us to solve this problem in analytic form and even to obtain exact formulas.

We now present an *exact formula*. For instance, the $(1,1)$-component of Λ_e has the form

$$\lambda_e^x = 1 + 2\rho c Re \sum_{m=0}^{\infty} A_m\left(r^2\right)\left(\rho r^2\right)^m, \tag{1}$$

where r is the radius of cylinders,

$$\begin{aligned} A_1(x) &= \alpha^{-1} 2\zeta(\alpha/2),\ A_2(x) = \sum_{n=1}^{\infty} \sigma_n^1 S_{2n} x^{2(n-1)}, \\ A_m(x) &= \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\ldots\sum_{n_{m-1}=1}^{\infty} \sigma_{n_1}^{n_2}\sigma_{n_2}^{n_3}\ldots\sigma_{n_{m-2}}^{n_{m-1}}\sigma_{n_{m-1}}^{1} S_{2n_1} x^{2(n_1+\ldots n_{m-1}-m+1)} \end{aligned}$$

$\zeta(z)$ is the Weirstrass elliptic function, $\mathbf{S}_q(z) = \sum_{n=1}^{\infty} \sigma_n^q\ z^{2(n-1)} = E_q(z) - z^{-q}$ $(q = 1, 2, ...)$, $E_q(z)$ are the Eisenstein functions represented by the Laurent expansions, $S_{2n} := \mathbf{S}_q(0)$. For details see [Mityushev V. Steady heat conduction of the material with an array of cylindrical holes in the non-linear case, IMA J Appl. Math., 61, 91-102, 1998] and [Mityushev V. Exact solution of the **R**–linear problem for a disk in a class of doubly periodic functions (Proc. Workshop on Complex Analysis) to appear]

An effective algorithm is constructed to calculate the effective conductivity of the composites with many different circular inclusions in the unit cell. The final formula for the effective conductivity tensor involves locations of the centers of inclusions, conductivities of constitutes and radii of inclusions in analytical form. For details see [Berlyand L., Mityushev V. Generalized Clausius-Mossotti formula for random composite with circular fibers, J Statist. Phys. v.102, N 1/2, 2001, 115-145], [Mityushev V. Transport properties of doubly periodic arrays of circular cylinders and optimal design problems, Appl.Math.& Optimization, 2001, v.44, 17-31], [Szczepkowski J., Malevich A. E., Mityushev V., Macroscopic properties of similar arrays of cylinders, Quart J Appl Math Mech, 56, 617-628, 2003] and [Berlyand L., Mityushev V. Increase and Decrease of the Effective Conductivity of Two Phase Composites due to Polydispersity, J Statist. Phys., v. 118, N 3-4, 481 - 509, 2005]. The effective conductivity tensor for non-linear composite materials is also discussed.

Longitudial *permeability* of a doubly periodic rectangular array of circular cylinders is studied. Analytical formulas analogous to (1) are deduced. For details see [Mityushev V., Adler P.M., Longitudial permeability of a doubly periodic rectangular array of circular cylinders, I, ZAMM, 82, N 5, 2002, 335-345] and [Mityushev V., Adler P.M., Longitudial permeability of a doubly periodic rectangular arrays of circular cylinders, II, ZAMP, 53, 2002, 486-517].

We consider *diffusion* on rough and spatially periodic surfaces. The macroscopic diffusion tensor D is determined by averaging the local fluxes over the unit cell. In [G. Pólya and G. Szegö. Isoperimetric Inequalities in Mathematical Physics, Princeton University Press, 1951] D is presented as the surface capacity. D is proved to be the unit tensor for macroscopically isotropic surfaces. For general surfaces, an asymptotic analysis is applied, when the ratio of the oscillation amplitude to the size of the unit cell is a small parameter ε. The microscopic field is determined up to $O(\varepsilon^2)$ in analytical form and an algorithm is derived to calculate higher order terms. We also deduce general analytical formulae for D up to $O(\varepsilon^4)$ and derive an algorithm to compute D as a series in ε^2. For details see [Adler P., Malevich A.E., Mityushev V., Macroscopic diffusion on a class of surfaces, Physical Review E, 69, 011607, 2004].

Electrokinetic phenomena in porous media are studied by application of the effective medium theory and the theory of duality transformation. We deduce new exact relations and analytical formulas for the effective constants of the macroscopic tensor. We also prove that the effective tensors obtained by these approaches coincide for 2D problems. The obtained results for the electrokinetic processes are closely related to similar results derived for piezoelectric composites because of a common mathematical background. For details see [Adler P.M., Mityushev V., Effective medium approximations and exact formulas for electrokinetic phenomena in porous media, J. Phys. A., Math. Gen. 36, 391-404, 2003].

Consider the flow through a three-dimensional curvilinear channel described by Stokes or *Navier-Stokes equations*. The amplitudes of the oscillations are proportional to a dimensionless parameter ε. The application of an analytical-numerical algorithm yields efficient formulas include ε^n in symbolic form. When ε increases, the Poiseuille flow ($\varepsilon = 0$) is disturbed and eddies can arise from a critical value $\varepsilon = \varepsilon_{crit}$. For details see [Adler P., Malevich A.E., Mityushev V. to appear in Acta Mechanica]